PROJECT #10: COMPUTATION OF LINEARIZED EULER EQUATIONS

This project solves the linearized Euler equations for flow past a thin airfoil. Flow is assumed to be uniform ($\rho = 1, u = M_\infty, v = 0$) at inflow and is used as the reference state for the local linearization.
We also simplify the equations by assuming constant temperature, i.e. Pressure = ρ.

Equations: The Euler equations for flow about a slender body in two dimensions can be written as (before linearization)

$$\frac{\partial Q}{\partial t} + \frac{\partial E(Q)}{\partial x} + \frac{\partial F(Q)}{\partial y} = 0 \tag{1}$$

with

$$Q = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, E = \begin{pmatrix} \rho u \\ \rho u^2 + \rho \\ \rho uv \end{pmatrix}, \quad F = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + \rho \end{pmatrix} \tag{2}$$

Note: Q, E, F *are* 3×1 *vectors.*
We rewrite in quasi-linear form, using e.g.

$$\frac{\partial E}{\partial x} = \frac{\partial E}{\partial Q}\frac{\partial Q}{\partial x} = A\frac{\partial Q}{\partial x}$$

gives us

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{3}$$

with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u^2 + 1 & 2u & 0 \\ -uv & v & u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ -uv & v & u \\ -v^2 + 1 & 0 & 2v \end{pmatrix} \tag{4}$$

To simplify the application we freeze A and B at the reference state $\rho = 1, u = M_\infty, v = 0$ to give us

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} + B\frac{\partial Q}{\partial y} = 0 \tag{5}$$

now with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -M_\infty^2 + 1 & 2M_\infty & 0 \\ 0 & 0 & M_\infty \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & M_\infty \\ 1 & 0 & 0 \end{pmatrix} \tag{6}$$

Now we are working with a small disturbance form of the Euler equations where ρ, u and v are the pertubation components from a uniform flow in the x direction. The Mach number is M_∞ and the equations can be used to study subsonic to supersonic small disturbance flow over slender bodies or past surfaces with small surface variations. The matrix A has **real** distinct eigenvalues $M_\infty, M_\infty + 1, M_\infty - 1$ and B the eigenvalues $0, 1, -1$, so that the system is hyperbolic in time.

Geometry: The grid is uniform in $-1 \le x \le 3.0$ and $0 \le y \le 2$, although you can choose any limits you want. At the lower surface, a biconvex thin airfoil is used with

$$y_{wall} = \tau x(1-x)/2 \quad 0 \le x \le 1 \tag{7}$$

$$y_{wall} = 0 \quad x \le 0, x \ge 1 \tag{8}$$

where τ is the thickness.
Boundary conditions:

1. At inflow ($x = -1$), fix $\rho u = M_\infty, \rho v = 0$, and set $\frac{\partial \rho}{\partial x} = 0$.
2. At the top ($y = 2$) fix all the variables $\rho = 1, u = M_\infty, v = 0$.
3. At outflow, ($x = 3$), use $\frac{\partial Q}{\partial x} = 0$.
4. Assume that v is specified at the lower boundary ($y = 0$) in x using thin airfoil conditions, that is

$$v = M_\infty \frac{dy_{wall}}{dx} \quad \text{imposed at} \quad y = 0 \tag{9}$$

Project 8 Apply the Euler Explicit and RK3 Time Differencing and Flux Splitting Method in space.
First starting with, the Euler explicit time differening ($h = \Delta t$), we have

$$Q^{(n+1)} = Q^{(n)} + hR(Q)^{(n)} \tag{10}$$

where

$$R(Q) = -A\frac{\partial Q}{\partial x} - B\frac{\partial Q}{\partial y}$$

Discretize the field using a uniform grid with $x_{j,k} = (j-1)\Delta x$ and $y_{j,k} = (k-1)\Delta y$. The matries A, B can be $\pm$ flux split into A^+, B^+ and A^-, B^- as discussed in class. This produces the new system to be solved with

$$R(Q)^{(n)}_{j,k} = -A^+\delta^b_x Q^{(n)}_{j,k} - A^-\delta^f_x Q^{(n)}_{j,k} - B^+\delta^b_y Q^{(n)}_{j,k} - B^-\delta^f_y Q^{(n)}_{j,k} \tag{11}$$

where e.g., δ^b_x is a backward differencing operator and δ^f_x is a forward differencing operator. Equation 10 along with Eq. 11 can be integrated from the uniform initial condition $Q^(0) = \begin{pmatrix} 1 \\ M_\infty \\ 0 \end{pmatrix}$ to a steady state.

A sample Matlab code is provided for you. It uses Euler explicit time differencing with central space differencing. It is unstable for all CFL, but will run for awhile at low CFL. Try $CFL = 0.1$ and $nx = 10$, (nx is used to define the grid in x and y). I put nx points on the airfoil, then $\Delta x = 1.0/(nx-1)$, let $\Delta y = \Delta x$ and compute the total number of points for the problem. I suggest you use this code as a starting point. It produces plots of $C_p = (\rho - 1)/(0.5 * M^2_\infty)$ at the wall, some density contours, and residual history, $||R||_2$.

Assignment

1. Program the Euler Explicit scheme Eq. 10 with the Flux Split form, Eq. 11 for a uniform grid on the domain ($-1 \leq x \leq 3$, $0 \leq y \leq 2$) using $\tau = 0.1$. Use 1^{st} and 2^{nd} order one-sided differences for $\delta_x^b, \delta_x^f, \delta_y^b$, and δ_y^f.

2. Replace the Euler Explicit scheme with Third-Order Runge-Kutta and compare the performance and other aspects discussed below.

   (a) The RK3 scheme is defined as

$$\begin{aligned} \widehat{Q} &= Q^{(n)} + \tfrac{1}{4} hR(Q)^{(n)} \\ \tilde{Q} &= Q^{(n)} + \tfrac{8}{15} hR(Q)^{(n)} \\ \overline{Q} &= \widehat{Q} + \tfrac{5}{12} hR(\tilde{Q}) \\ Q^{(n+1)} &= \widehat{Q} + \tfrac{3}{4} hR(\overline{Q}) \end{aligned} \tag{12}$$

   (b) **Note: the proper use of $\widehat{Q}, \tilde{Q}, \overline{Q}$ is very important!**

3. Find the expression for the spatial accuracy of this method, i.e., what is er_t or what is the modified wave number? *Hint* Remember, we are working with a linear coupled system. To do the analysis you need to think in terms of the decouple representative equations.

4. Find $\sigma - \lambda$ relation for the RK3 $O\Delta E$ scheme

5. Using the $\sigma - \lambda$ relations for Euler Explicit and Rk3, Come up with a stability condition and convergence estimates for different differencing orders. One possible CFL definition for this system is

$$CFL = \frac{h\ (max(\lambda(A)) + max(\lambda(B)))}{\min(dx, dy)}$$

6. Study various CFL numbers (remembering the numerical stability condition!), to see if your analysis is consistent with your results.

Suggestions, Questions:

1. You can play around with the value of M_∞. Try $M_\infty > 1.0$.

2. Start using $nx = 10$ and do a grid refinement study.

3. You should play around with the CFL. Why? Is there an optimal value of CFL for accurate results and the best convergence?

4. What happens when you violate the stability condition? Try it.

5. The boundary conditions supplied are rather crude. Try to develop a better set, implement them and check their performance, effect on stability and convergence.

6. **Note: Important** You will notice that since we do not start with the exact solution, that you may get glitches in the initial responses and values of C_p. Don't let this hold you up.

General Instructions:
Follow the instructions given above and address each of the assignments. You will need to provide me with a **short** writeup of what you have done, along with some results and figures. This can be handwritten, but I prefer TeX, LaTeX or some other word processor form. Perform all the computations using MATLAB. I will also want copies of all the source codes. (You will be required to email them to me, I will make arrangements).
There will be 10 minutes allotted for a short presentation in class on what you have accomplished. You should focus on the interesting aspects of your project.
This project will account for 50% of your grade. I will be judging it on the write-up, a working code (I will run all the codes you send me), and your presentation. Try to focus on some key results in the presentations. I don't wany a lot of slides and repetition.
Remember, **10 Minute Presentations**